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1980 J. Phys. A: Math. Gen. 13 569

(http://iopscience.iop.org/0305-4470/13/2/023)

J. Phys. A: Math. Gen., **13** (1980) 569–584. Printed in Great Britain

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Received 19 July 1979

Abstract. As an extension of previous work which developed a method for performing calculations in curved space–time quantum field theory using momentum space, it is shown how the technique can be used to great advantage in the computation of both the divergent and finite parts of S-matrix elements for self-interacting $\lambda\phi^4$ theory. Using dimensional regularisation all the pole terms in the self-energy up to second order are calculated, to confirm that, to this order, mass-independent renormalisation can be successfully applied.

1. Introduction

In a recent paper (Birrell 1979a) a method of performing quantum field theory calculations in curved space–time using momentum space was presented. The motivation for the development of this method was that it allows an efficient means of performing calculations which were previously not feasible. To illustrate this it was shown how the method could easily be used to calculate the number of particles produced from a free field in an expanding universe. It was also mentioned how the Feynman propagator for a free field could be calculated using the momentum space technique and that this would be of value in determining the energy momentum tensor for the field, or the effects of interactions. Although in the introductory paper (Birrell 1979a) no examples of such calculations were given, it is in these areas that the momentum space method is most valuable.

The reasons for wishing to calculate the energy momentum tensor, or the effects of self-interactions of a field in curved space–time, are numerous. The energy momentum tensor, via the Einstein field equations, determines the evolution of the space–time. It is also imperative that we should consider interacting fields rather than free fields since it is the former which have observable effects in nature apart from coupling to gravity. In this paper it will be shown how the momentum space method can be used to facilitate calculations involving self-interacting, scalar fields in curved space–time. In passing, the property of the method which also makes it so useful for energy momentum tensor calculations will be pointed out. Such energy momentum tensor calculations will be described in a future paper.

Preliminary estimates of the effects of self-interactions on particle production have been made in a paper by Birrell and Ford (1978) where it was found that such effects could be significant compared with the production of free particles by the means described in the previous paper on the momentum space method (Birrell 1979a).

Unfortunately these estimates were hampered by calculational difficulties and were restricted to a small number of situations. However, the type of difficulties encountered were of just the type that the momentum space method was designed to solve, as shall be made evident in the next section. In particular, it was found not to be possible to make much progess without resorting to numerical calculations, and even then only with considerable difficulty. One of the strong points of the momentum space method described previously (Birrell 1979a) is that it facilitates numerical work, allowing the adoption of techniques long used in nuclear physics.

In turning to interacting quantum field theory, one is faced with the problem of ultraviolet divergences, just as in flat space. A detailed, general analysis of the ultraviolet divergences arising in self-interacting quantum field theories in curved space–time has been undertaken by Birrell and Taylor (1978) who find, using Green function diagram techniques and the theory of the products of distributions, that the overall divergence structure is the same as in flat space, and that all the infinities can be removed by renormalisation of constants appearing in the Lagrangian. This is provided that it can be shown that all the infinities arising from overlapping divergences cancel, or can be made to cancel by a suitable modification to the theory. If this is not the case, then, as has been pointed out in a recent paper of Bunch *et al* (1979), there can arise non-geometrical, state-dependent infinities, which cannot be absorbed into existing constants in the Lagrangian. Bunch *et al* (1979) and T S Bunch (private communication) note that they can show that in a conformally flat space–time all such state-dependent infinities cancel to second order in perturbation theory in $\lambda\phi^4$ theory (although that this is generally the case is not clear (see Birrell and Taylor (1978)).

The main aim of this paper will be to show how the momentum space method handles ultraviolet divergences, as well as setting the stage for calculation of the interesting finite parts of S-matrix elements that will play an important role in cosmological or astrophysical processes, and which will be discussed elsewhere (Birrell *et al* 1980). We shall see that the divergences can be handled independently of the space–time under consideration, leaving the finite part of the S-matrix elements to be calculated without need for any regularisation scheme, thus considerably simplifying their numerical computation.

In the next section relevant parts of the momentum space method will be reviewed, with particular emphasis on the calculation of the Feynman propagator. In § 3 it will be shown how the method can be used to calculate the divergences in $\lambda\phi^4$ theory to second order, and that the parts of the S-matrix which depend explicitly on the space–time under consideration are all finite (apart from some terms proportional to the scalar curvature). The final section concludes by mentioning desirable future applications of the scheme outlined in §§ 2 and 3.

2. Review of the momentum space method

We review and extend to n-dimensional space–times, the relevant parts of Birrell (1979a) in the framework of a scalar field with $\lambda\phi^4$ self-interaction as discussed in Birrell and Ford (1978) or Bunch *et al* (1979). We confine our attention to a conformally flat space–time with metric

$$ds^2 = \Omega^2(\eta)\left[d\eta^2 - \sum_{i=1}^{n-1} (dx^i)^2\right] \tag{2.1}$$

although extension to certain other metrics would not be difficult as was mentioned in Birrell (1979a) (where the limitations of the momentum space method in this respect were also discussed). The scalar field has Lagrangian density

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I \tag{2.2}$$

where

$$\mathscr{L}_0 = \tfrac{1}{2}\sqrt{-g}[g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - (m^2 + \xi R)\phi^2] \tag{2.3}$$

$$\mathscr{L}_I = \sqrt{-g}\Big[-\frac{1}{2}(\delta m^2 + \delta\xi R)\phi^2 - \frac{1}{4!}\lambda_B\phi^4\Big] \tag{2.4}$$

with m and ξ the renormalised mass and conformal coupling constant respectively, and λ_B the bare interaction coupling constant. We note that the notation used here is slightly different from that of Birrell (1979) in that ξ (this paper) $= (\xi + \frac{1}{6})$ (Birrell 1979) in four dimensions. This is for conformity with the papers on interacting field theory in curved space–time mentioned above (we use Misner *et al* (1973) $(---)$ sign conventions).

In the next section we wish to calculate Feynman diagrams which are written down in terms of the Feynman propagator $G_F(x, x')$, which satisfies (see Birrell 1979a,b)

$$\sqrt{-g}[\Box_x + \xi R(x) + m^2]G_F(x, x') = -\delta^n(x - x'). \tag{2.5}$$

Defining

$$g_F(x, x') = [\Omega(x)]^{(n-2)/2} G_F(x, x')[\Omega(x')]^{(n-2)/2} \tag{2.6}$$

equation (2.5) gives

$$[\tilde{\Box}_x + m_-^2]g_F(x, x') = -\delta^n(x - x') + V(\eta)g_F(x, x') \tag{2.7}$$

where $\tilde{\Box}_x$ is the flat space–time D'alembertian, $\partial_\mu\,\partial^\mu$, and we have defined

$$V(\eta) = (m_-^2 - m^2\Omega^2(\eta)) - \Big(\xi - \frac{n-2}{4(n-1)}\Big)\Omega^2(\eta)R(\eta) \tag{2.8}$$

with

$$m_- = m\Omega(\eta = -\infty) \tag{2.9}$$

(the removal of the restriction $\Omega(-\infty) < \infty$ implied by (2.9) is discussed in § 7 of Birrell (1979a)).

If we let $\tilde{G}_F(x, x')$ be the flat space–time Feynman propagator, given in momentum space by

$$\tilde{G}_F(x, x') = (2\pi)^{-n} \int e^{ip(x-x')}\,\tilde{G}_F(p_0; |\boldsymbol{p}|)\,d^n p \tag{2.10}$$

with

$$\tilde{G}_F(p_0; |\boldsymbol{p}|) = (p_0^2 - |\boldsymbol{p}|^2 - m_-^2 + i\epsilon)^{-1} \tag{2.11}$$

then it was shown in Birrell (1979a) that

$$g_F(x, x') = (2\pi)^{-n} \int e^{-i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{x}')}\,e^{ip_0\eta}g_F(p_0, p_0'; |\boldsymbol{p}|)\,e^{-ip_0'\eta'}\,d\boldsymbol{p}\,dp_0\,dp_0' \tag{2.12}$$

where

$$g_F(q, q'; k) = \delta(q - q')\tilde{G}_F(q; k) - \tilde{G}_F(q; k)t_k(q, q')\tilde{G}_F(q'; k) \tag{2.13}$$

and the 't matrix' t_k satisfies a Lippmann–Schwinger type equation

$$t_k(p, p') = V(p, p') - \int_{-\infty}^{\infty} \tilde{G}_F(q; k) \tilde{V}(p, q) t_k(q, p') \, dq \tag{2.14}$$

$\hat{V}(p, q)$ being the Fourier transform of $V(\eta)$:

$$\hat{V}(p, q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(p-q)\eta} V(\eta) \, d\eta \tag{2.15}$$

which will also be written as $\hat{V}(p-q)$. Substituting (2.14) into (2.13) gives the form of g_F which will be used in the next section:

$$g_F(q, q'; k) = \sum_{i=1}^{3} g_F^{(i)}(q, q'; k) \tag{2.16}$$

where

$$g_F^{(1)}(q, q'; k) \equiv \delta(q - q') \tilde{G}_F(q; k) \tag{2.17a}$$

$$g_F^{(2)}(q, q'; k) \equiv -\tilde{G}_F(q; k) \hat{V}(q, q') \tilde{G}_F(q'; k) \tag{2.17b}$$

$$g_F^{(3)}(q, q'; k) = \tilde{G}_F(q, k) \tilde{G}_F(q'; k) \int_{-\infty}^{\infty} \tilde{G}_F(p; k) \hat{V}(q - p) t_k(p, q') \, dp. \tag{2.17c}$$

The corresponding coordinate space $g^{(i)}$ are obtained by substituting (2.17) into (2.12). $g_F^{(1)}$ and $g_F^{(2)}$ are, respectively, the first and second Born approximations to g_F.

It is at this point that the advantage of using this method in calculating the energy momentum tensor for a free field becomes evident. It is not difficult to verify that all the divergences which arise in calculating the energy momentum tensor arise from $g_F^{(1)}$ and $g_F^{(2)}$ in a massless theory, and from these, and the third Born approximation term, obtained by putting $\hat{V}(p-q)$ for $t_k(p, q)$ in $g_F^{(3)}$, in the massive case. Thus it is only necessary to regularise and renormalise these lowest order Born contributions, leaving a finite remainder involving the t matrix, which can be calculated numerically, for example. In fact Davies and Unruh (1979) have calculated, by coordinate space methods, the renormalised energy momentum tensor for a massless field resulting from $g_F^{(1)}$ and $g_F^{(2)}$, thus taking care of all the infinities. To find the complete stress tensor for a given space–time, one now only needs to solve (2.14) for t_k, substitute it into (2.17c), the result of which goes into (2.12) to be differentiated to form the remainder of the stress tensor which is added to Davies and Unruh's result. Although this may sound complicated, since all the infinities have already been taken care of one is only dealing with finite quantities, making the entire procedure most amenable to numerical computation. It is hoped to publish details in the near future.

It is for reasons similar to those just discussed for the stress tensor that make the momentum space method so advantageous for use in interacting field calculations. It is to these that we now turn.

3. Perturbation calculations in momentum space

Since nearly all flat space–time calculations of Feynman diagrams are carried out in momentum space, we should hope to find some advantage in mimiking these calculations in curved space–time. This indeed turns out to be the case, and in this section we shall repeat in curved space–time a calculation performed by Collins (1974) in flat space–time. Using dimensional regularisation, and the 't Hooft (1972) mass-independent renormalisation scheme, Collins calculated the pole terms of all the vertex and self-energy diagrams up to second order in $\lambda\phi^4$ theory, showing explicitly that mass-independent renormalisation does work. In curved space–time, showing that a mass-independent renormalisation scheme can be used is even more important than in flat space–time, as this is equivalent to showing that the state-dependent infinities mentioned in the introduction do in fact cancel. It is also necessary for this to be the case if the desire to calculate the majority of the space–time depedent parts of the S matrix without the need to worry about infinities and regularisation is to be realised.

The mass-independent renormalisation procedure is developed as in flat space–time ('t Hooft 1973, Collins 1974). With the additional possibility of ξ being renormalised we write

$$m_B^2 = m^2 + \delta m^2 = m^2 + \sum_{\nu=1}^{\infty} \sum_{j=\nu}^{\infty} \frac{m^2 b_{\nu j} \lambda_R^j}{(n-4)^\nu} \tag{3.1a}$$

$$\xi_B = \xi + \delta\xi = \xi + \sum_{\nu=1}^{\infty} \sum_{j=\nu}^{\infty} \frac{d_{\nu j} \lambda_R^j}{(n-4)^\nu} \tag{3.1b}$$

$$\lambda_B = \mu^{4-n}\left[\lambda_R + \sum_{\nu=1}^{\infty} \sum_{j=\nu}^{\infty} \frac{a_{\nu j} \lambda_R^j}{(n-4)^\nu}\right] \tag{3.2}$$

$$Z(\lambda_R, n) = 1 + \sum_{\nu=1}^{\infty} \sum_{j=\nu}^{\infty} \frac{c_{\nu j} \lambda_R^j}{(n-4)^\nu} \tag{3.3}$$

the final quantity being the field renormalisation constant. The special problems associated with defining S-matrix elements in curved space–time have been discussed in the references given earlier (Birrell and Taylor 1978, Birrell and Ford 1978, Bunch *et al* 1979) and will not be repeated here. We merely note that once the problem of choosing 'in' and 'out' vacuum states has been handled one can write down expressions for physical quantities of interest in terms of matrix elements between 'in' states.

Working in the interaction picture we decompose the field ϕ as

$$\phi(x) = \int \mathrm{d}^{n-1}k\,(a_k F_k + \mathrm{HC}) \tag{3.4}$$

then at order m in perturbation theory the S matrix receives a contribution (for $m > 0$)

$$S^{(m)} = \frac{\mathrm{i}^m}{m!} \int T(\mathcal{L}_I(x_1)\mathcal{L}_I(x_2) \ldots \mathcal{L}_I(x_m))\, \mathrm{d}^n x_1 \ldots \mathrm{d}^n x_m. \tag{3.5}$$

It is a straightforward task to obtain coordinate space expressions for all the connected, one-particle irreducible components of $\langle k_1|S^{(n)}|k_2\rangle$ to second order (these being the components contributing to the self-energy), where by $\langle k_1|$ we mean ${}_{\mathrm{in}}\langle \boldsymbol{k}_1|$ in the notation of Birrell and Ford (1978). We find

$$\langle k_1|S^{(1)}|k_2\rangle = \tfrac{1}{2}\lambda_B \int \sqrt{-g_x}\, G_{\mathrm{F}}(x, x) F_{k_1}^*(x) F_{k_2}(x)\, \mathrm{d}^n x$$
$$- \mathrm{i} \int \sqrt{-g_x} (\delta m^2 + \delta\xi R(x)) F_{k_1}^*(x) F_{k_2}(x)\, \mathrm{d}^n(x) \tag{3.6}$$

and

$$\langle k_1|S^{(2)}|k_2\rangle^{(\mathrm{conn.,opi})}$$
$$=\tfrac{1}{2}\lambda_B\int\int\sqrt{-g_x}\sqrt{-g_y}(\delta m^2+\delta\xi R(y))G_{\mathrm{F}}^2(x,y)F_{k_1}^*(x)F_{k_2}(x)\,\mathrm{d}^nx\,\mathrm{d}^ny$$
$$+\frac{\mathrm{i}}{4}\lambda_B^2\int\int\sqrt{-g_x}\sqrt{-g_y}G_{\mathrm{F}}^2(x,y)G_{\mathrm{F}}(y,y)F_{k_1}^*(x)F_{k_2}(x)\,\mathrm{d}^nx\,\mathrm{d}^ny$$
$$+\frac{\mathrm{i}}{6}\lambda_B^2\int\int\sqrt{-g_x}\sqrt{-g_y}G_{\mathrm{F}}^3(x,y)F_{k_1}^*(x)F_{k_2}(y)\,\mathrm{d}^nx\,\mathrm{d}^ny. \tag{3.7}$$

Equation (3.6) corresponds to figure 1(a) of Collins (1974), plus its counterterm, while the first, second and third lines of (3.7) give his figures 1(c), 1(b) and 1(d), respectively. To determine the relation between the bare and renormalised coupling constants we shall also need the connected, one-particle irreducible, four-particle S-matrix elements to second order:

$$\langle k_1,k_2|S^{(1)}|k_3,k_4\rangle=-\mathrm{i}\lambda_B\int\sqrt{-g_x}F_{k_1}^*(x)F_{k_2}^*(x)F_{k_3}(x)F_{k_4}(x)\,\mathrm{d}^nx \tag{3.8}$$

$$\langle k_1,k_2|S^{(2)}|k_3,k_4\rangle_{(\mathrm{conn.,opi})}$$
$$=\tfrac{1}{2}\lambda_B^2\int\int\sqrt{-g_x}\sqrt{-g_y}G_{\mathrm{F}}^2(x,y)[F_{k_1}^*(x)F_{k_2}^*(x)F_{k_3}(y)F_{k_4}(y)$$
$$+F_{k_1}^*(x)F_{k_2}^*(y)F_{k_3}(x)F_{k_4}(y)+F_{k_1}^*(y)F_{k_2}^*(x)F_{k_3}(x)F_{k_4}(y)]\,\mathrm{d}^nx\,\mathrm{d}^ny \tag{3.9}$$

which corresponds to Collin's figure 2.

We now wish to consider the infinities generated by (3.6)–(3.9) and show how they are removed by the various counterterms. Since we are only interested in the renormalisation procedure we shall not need explicit expressions for the modes $F_k(x)$. However, when it comes to calculating finite parts of the S-matrix elements (Birrell *et al* 1980), these are needed and are calculated in a straightforward manner using the momentum space formulation (Birrell 1979a). In fact, in what follows, rather than working with the S-matrix elements themselves we shall consider $\Pi(x,x')$, the self-energy, defined by (see Birrell and Taylor 1978)

$$\langle k_1|S|k_2\rangle-\delta_{k_1k_2}=\int\sqrt{-g_x}\sqrt{-g_{x'}}F_{k_1}^*(x)F_{k_2}(x')\Pi(x,x')\,\mathrm{d}^nx\,\mathrm{d}^nx'. \tag{3.10}$$

To perform the calculation, all we need is the Feynman propagator in n-dimensions, and we have seen how to determine this in momentum space in the previous section†.

From (3.6), (3.1) and (3.2), we have a contribution to the self-energy to first order which we shall denote $\Pi^{(1,1)}$ and which is given by

$$\Pi^{(1,1)}(x,x')=\left[\frac{1}{2}\mu^{4-n}\lambda_RG_{\mathrm{F}}(x,x)-\frac{\mathrm{i}\lambda_R}{(n-4)}(m^2b_{11}+d_{11}R(x))\right]\frac{\delta^n(x-x')}{\sqrt{-g_{x'}}}. \tag{3.11}$$

† One can show that the propagator of the previous section is in fact an ⟨out|. . .|in⟩ expectation value which should not strictly be used in the calculation of ⟨in|. . .|in⟩ S-matrix elements. However, the equations for the self-energy are the same in either set of states; only the external wavefunctions are different. This then is a subtlety which need not concern us here; it is discussed fully in Birrell (1979b).

We thus need to calculate $G_F(x, x)$, which we write in analogy to (2.16) as

$$G_F(x, x) = \sum_{i=1}^{3} G_F^{(i)}(x, x) \tag{3.12}$$

and find, using (2.6), (2.17a), (2.12) and (2.11), that

$$\begin{aligned} G_F^{(1)}(x, x) &= (2\pi)^{-n}[\Omega(\eta)]^{2-n} \int \mathrm{d}^n p (p^2 - m_-^2 + \mathrm{i}\epsilon)^{-1} \\ &= -\mathrm{i}(4\pi)^{-n/2}(m_-^2)^{(n/2)-1}[\Omega(\eta)]^{2-n}\Gamma\left(1-\frac{n}{2}\right) \\ &= -\mathrm{i}(4\pi)^{-n/2}(m_-^2)^{(n/2)-1}[\Omega(\eta)]^{2-n}\left(\frac{2}{n-4}+\gamma-1+\mathrm{O}(n-4)\right). \end{aligned} \tag{3.13}$$

In performing the integral we have used equation (A.1) of Collins (1974). Next, using (2.6), (2.17*b*) and (2.12) we have

$$G_F^{(2)}(x, x) = (2\pi)^{-n}[\Omega(\eta)]^{2-n} \int \frac{\mathrm{e}^{\mathrm{i}(p_0 - p_0')\eta}\hat{V}(p_0 - p_0')}{(p^2 - m_-^2)(p'^2 - m_-^2)} \mathrm{d}p_0' \, \mathrm{d}^n p \tag{3.14}$$

where $p' = (p_0', \boldsymbol{p}')$, and we have suppressed the writing of $\mathrm{i}\epsilon$. Changing from integration variable p_0' to $k_0 = p_0 - p_0'$, introducing a Feynman parameter and performing the n-dimensional integral we find

$$\begin{aligned} G_F^{(2)}(x, x) &= -\mathrm{i}(4\pi)^{-n/2}[\Omega(\eta)]^{2-n}\Gamma\left(2-\frac{n}{2}\right)\int_{-\infty}^{\infty} \mathrm{d}k_0 \, \mathrm{e}^{\mathrm{i}k_0\eta}\hat{V}(k_0) \\ &\quad \times \int_0^1 \mathrm{d}x[m_-^2 - k_0^2 x(1-x)]^{(n/2)-2} \\ &= 2\mathrm{i}(4\pi)^{-n/2}[\Omega(\eta)]^{2-n}\frac{1}{(n-4)}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_0\eta}\hat{V}(k_0)\,\mathrm{d}k_0 \\ &\quad + \mathrm{i}(4\pi)^{-n/2}[\Omega(\eta)]^{2-n}\int_{-\infty}^{\infty} \mathrm{d}k_0 \, \mathrm{e}^{\mathrm{i}k_0\eta}\hat{V}(k_0) \\ &\quad \times \left\{\gamma + \int_0^1 \mathrm{d}x \ln[m_-^2 - k_0^2 x(1-x)]\right\} + \mathrm{O}(n-4). \end{aligned} \tag{3.15}$$

The final component of G_F comes from (2.17*c*):

$$G_F^{(3)}(x, x) = (4\pi^2)^{-n/2}[\Omega(\eta)]^{2-n} \int \frac{\mathrm{e}^{\mathrm{i}(p_0 - p_0')\eta}\hat{V}(p_0 - q_0)t_{|\boldsymbol{p}|}(q_0, p_0')\,\mathrm{d}^n p\,\mathrm{d}p_0'\,\mathrm{d}q_0}{(p^2 - m_-^2)(q^2 - m_-^2)(p'^2 - m_-^2)}$$

where $p' \equiv (p_0', \boldsymbol{p})$, $q \equiv (q_0, \boldsymbol{p})$. We see by power counting that this component will not have a pole at $n = 4$ (for details see Birrell 1979b).

Some manipulation using the inverse of (2.16) and definition (2.8) allows us to write

$$G_F(x, x) = G_F^{\text{pole}}(x, x) + G_F^{\text{finite}}(x, x) \tag{3.16}$$

where

$$G_F^{\text{pole}}(x, x) = -\frac{2\mathrm{i}}{(4\pi)^2}\frac{[m^2 + (\xi - \frac{1}{6})R]}{(n-4)} \tag{3.17}$$

$$
\begin{aligned}
G_{\mathrm{F}}^{\mathrm{finite}}(x,) = & -\frac{\mathrm{i}}{(4\pi)^2}[m^2+(\xi-\tfrac{1}{6})R]\left[\gamma-1+\ln\left(\frac{m_-^2}{4\pi\Omega(\eta)^2}\right)\right] \\
& +\frac{\mathrm{i}}{(4\pi)^2}\frac{R}{18}+\frac{\mathrm{i}\Omega^{-2}(\eta)}{(4\pi)^2}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k_0\eta}\hat{V}(k_0) \\
& \times\left\{1+\int_0^1\ln\left[1-\left(\frac{k_0}{m_-}\right)^2 x(1-x)\right]\mathrm{d}x\right\}\mathrm{d}k_0 \\
& +\frac{\Omega^{-2}(\eta)}{(4\pi^2)^2}\int\frac{\mathrm{e}^{\mathrm{i}(p_0-p_0')\eta}\hat{V}(p_0-q_0)t_{|\boldsymbol{p}|}(q_0,p_0')\,\mathrm{d}^n p\,\mathrm{d}p_0'\,\mathrm{d}q_0}{(p^2-m_-^2)(q^2-m_-^2)(p'^2-m_-^2)} \\
& +\mathrm{O}(n-4). \qquad (3.18)
\end{aligned}
$$

Upon substituting (3.18) and (3.11) (expanding $\mu^{4-n}=1+(4-n)\ln\mu+\mathrm{O}((4-n)^2)$), we see that the pole term can indeed be absorbed into mass and ξ renormalisations if we take

$$
\begin{aligned}
b_{11} &= -(16\pi^2)^{-1} \\
d_{11} &= -(\xi-\tfrac{1}{6})/(16\pi^2)^{-1} \qquad (3.19)
\end{aligned}
$$

which, in the case of b_{11}, is the same as obtained by Collins (1974) in flat space (there being no d_{ij} in flat space).

We next turn to the second-order calculation, starting with the vertex function (3.9). We have at second order

$$
\begin{aligned}
& \langle k_1, k_2|S^{(2)}|k_3, k_4\rangle \\
& \quad = \tfrac{1}{2}\mu^{8-2n}\lambda_R^2\int\sqrt{-g_x}\sqrt{-g_{x'}}\,G_{\mathrm{F}}^2(x,x')[F_{k_1}^*(x)F_{k_2}^*(x)F_{k_3}(x')F_{k_4}(x') \\
& \qquad + F_{k_1}^*(x)F_{k_2}^*(x')F_{k_3}(x)F_{k_4}(x')+F_{k_1}^*(x')F_{k_2}^*(x)F_{k_3}(x)F_{k_4}(x')]\,\mathrm{d}^n x\,\mathrm{d}^n x'. \qquad (3.20)
\end{aligned}
$$

From (2.6) and (2.12) we have

$$
G_{\mathrm{F}}^2(x,x')=[\Omega(\eta)\Omega(\eta')]^{2-n}I(x,x') \qquad (3.21)
$$

where

$$
\begin{aligned}
I(x,x') = (2\pi)^{-2n}\int & \mathrm{e}^{-\mathrm{i}\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{x}')}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{q}.(\boldsymbol{x}-\boldsymbol{x}')}\,\mathrm{e}^{\mathrm{i}(p_0\eta-p_0'\eta')}\,\mathrm{e}^{\mathrm{i}(q_0\eta-q_0'\eta')} \\
& \times g_{\mathrm{F}}(p_0,p_0';|\boldsymbol{p}|)g_{\mathrm{F}}(q_0,q_0';|\boldsymbol{q}|)\,\mathrm{d}^n p\,\mathrm{d}^n q\,\mathrm{d}p_0'\,\mathrm{d}q_0'. \qquad (3.22)
\end{aligned}
$$

Performing an obvious change of variables and using (2.16) we may write

$$
I(x,x')=(2\pi)^{-n}\sum_{i,j=1}^{3}\int\mathrm{e}^{-\mathrm{i}\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{x}')}\,\mathrm{e}^{\mathrm{i}(k_0\eta-k_0'\eta')}\,I^{(i,j)}(k,k_0')\,\mathrm{d}^n k\,\mathrm{d}k_0' \qquad (3.23)
$$

where

$$
I^{(i,j)}(k,k_0')=(2\pi)^{-n}\int g_{\mathrm{F}}^{(i)}(p_0,p_0';|\boldsymbol{p}|)g_{\mathrm{F}}^{(j)}(k_0-p_0,k_0'-p_0';|\boldsymbol{k}-\boldsymbol{p}|)\,\mathrm{d}^n p\,\mathrm{d}p_0'. \qquad (3.24)
$$

Next, using (2.17a), introducing a Feynman parameter and using Collins's (1974) equation (A.1) we find

$$I^{(1,1)}(k, k_0') = \mathrm{i}(4\pi)^{-n/2}\Gamma\left(2-\frac{n}{2}\right)\delta(k_0-k_0')\int_0^1 [m^2-k^2x(1-x)]^{(n/2)-2}\,\mathrm{d}x$$

$$= -\frac{2\mathrm{i}(4\pi)^{-n/2}}{(n-4)}\delta(k_0-k_0')+\text{finite}. \tag{3.25}$$

Use of power counting and the observation that there are no overlapping divergences in the diagram under discussion shows that all the other $I^{(i,j)}$ are finite at $n=4$ and thus do not contribute any pole terms. In a calculation of the finite parts of the S-matrix elements all the remaining $I^{(i,j)}$ would have to be calculated, but this can be done numerically, without regularisation using the formulae given above.

Substitution of (3.25) into (3.23) gives

$$I(x, x') = -\frac{2\mathrm{i}(4\pi)^{-n/2}}{(n-4)}\delta^n(x-x')+\text{finite}. \tag{3.26}$$

Using this in (3.21), which is then substituted in (3.20), expanded around $n=4$ using $\sqrt{-g}=\Omega^n$, gives

$$\langle k_1, k_2|S^{(2)}|k_3, k_4\rangle$$

$$= -\frac{3\mathrm{i}\lambda_R^2}{16\pi^2(n-4)}\int \sqrt{-g_x}F^*_{k_1}(x)F^*_{k_2}(x)F_{k_3}(x)F_{k_4}(x)\,\mathrm{d}^nx+\text{finite}. \tag{3.27}$$

The pole in (3.27) is cancelled by the pole arising from the substitution of (3.2) to second order into (3.8) if we take

$$a_{12} = -3(16\pi^2)^{-1} \tag{3.28}$$

once again in agreement with the result obtained by Collins (1974) in flat space–time.

We may now obtain the contribution to the self-energy coming from the substitution of (3.2) to second order into (3.6), but not including the δm^2 and $\delta\xi$ terms, which can conveniently be taken account of later by using the bare rather than the renormalised m and ξ in the inverse propagator in the right-hand side of equation (3.50), as was done by Collins (1974) in Minkowski space. We shall denote this as $\Pi^{(1,2)}$:

$$\Pi^{(1,2)}(x, x') = \{\tfrac{1}{2}\mu^{4-n}[\lambda_R+a_{12}\lambda_R^2(n-4)^{-1}]G_{\mathrm{F}}^{\mathrm{pole}}(x, x)$$

$$+\tfrac{1}{2}a_{12}\lambda_R^2G_{\mathrm{F}}^{\mathrm{finite}}(x, x)(n-4)^{-1}\}\frac{\delta^n(x-x')}{\sqrt{-g_{x'}}}$$

$$+\text{finite}. \tag{3.29}$$

The product of $a_{12}(n-4)^{-1}$ and $G_{\mathrm{F}}^{\mathrm{finite}}$ shows the first occurrence of state-dependent infinities of the type mentioned in the introduction. Unless all such terms cancel when all contributions to Π of second order are added together we shall not be able to use mass-independent renormalisation.

We next obtain the contribution to Π from the first two terms of (3.7). Denoting this contribution to second order as $\Pi^{(2,1)}$ we have

$$\Pi^{(2,1)}(x,x')=\lambda_R^2\int\sqrt{-g_y}G_F^2(x,y)$$
$$\times\Big(\tfrac{1}{2}\mu^{4-n}[(m^2b_{11}+d_{11}R(y))(n-4)^{-1}+\tfrac{1}{2}\mathrm{i}\mu^{4-n}G_F^{\text{pole}}(y,y)]$$
$$+\frac{\mathrm{i}}{4}\mu^{8-2n}G_F^{\text{finite}}(y,y)\Big)\,\mathrm{d}^n y\,\delta^n(x-x')/\sqrt{-g_{x'}}$$
$$=\tfrac{1}{4}\lambda_R^2\int\sqrt{-g_y}G_F^2(x,y)\{-(4\pi)^{-2}\ln(\mu^2)[m^2+(\xi-\tfrac{1}{6})R(y)]$$
$$+\mathrm{i}\mu^{8-2n}G_F^{\text{finite}}(y,y)\}\,\mathrm{d}^n y\,\delta^n(x-x')/\sqrt{-g_{x'}}+\text{finite} \tag{3.30}$$

where in obtaining the second equality we have used (3.17) and (3.19). We only need the pole part of G_F^2 which has already been calculated; it is obtained by substituting (3.26) into (3.21). We thus obtain

$$\Pi^{(2,1)}(x,x')=\frac{\lambda_R^2}{32\pi^2}\frac{1}{(n-4)}\Big(\mathrm{i}\frac{\ln(\mu^2)}{16\pi^2}[m^2+(\xi-\tfrac{1}{6})R]+G_F^{\text{finite}}(x',x')\Big)\frac{\delta^n(x-x')}{\sqrt{-g_{x'}}}+\text{finite}. \tag{3.31}$$

We now come to the only difficult part of the calculation, namely the contribution to the self-energy coming from the third line of (3.7). Denoting this as $\Pi^{(2,2)}$ we have to second order

$$\Pi^{(2,2)}(x,x')=\frac{\mathrm{i}}{6}\lambda_R^2\mu^{8-2n}G_F^3(x,x'). \tag{3.32}$$

Using (2.12) and (2.16) we have

$$\mu^{8-2n}G_F^3(x,x')=\mu^{8-2n}\sum_{i,j,l=1}^{3}G_F^{3(i,j,l)}(x,x') \tag{3.33}$$

where

$$\mu^{8-2n}G_F^{3(i,j,l)}(x,x')$$
$$=(2\pi)^{-3n}[\Omega(\eta)\Omega(\eta')]^{3(2-n)/2}\int \mathrm{e}^{\mathrm{i}\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{x}')}\,\mathrm{e}^{-\mathrm{i}(k_0\eta-k_0'\eta')}J^{(i,j,l)}(k,k_0)\,\mathrm{d}^n k\,\mathrm{d}k_0' \tag{3.34}$$

and

$$J^{(i,j,l)}(k,k_0')=\mu^{8-2n}\int g_F^{(i)}(p_0,p_0';|\boldsymbol{p}|)g_F^{(j)}(q_0,q_0';|\boldsymbol{q}|)$$
$$\times g_F^{(l)}(-p_0-q_0-k_0,-p_0'-q_0'-k_0';|\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{k}|)\,\mathrm{d}^n p\,\mathrm{d}^n q\,\mathrm{d}p_0'\,\mathrm{d}q_0'. \tag{3.35}$$

A change of variables has been used to put (3.34) in the form given. By considering various changes of variables it is easily verified that

$$G_F^3=G_F^{3(1,1,1)}+3G_F^{3(1,1,2)}+3G_F^{3(1,3,1)}$$
$$+G_F^{3(2,2,2)}+3G_F^{3(1,2,2)}+6G_F^{3(1,2,3)}+3G_F^{3(2,2,3)}$$
$$+3G_F^{3(1,3,3)}+3G_F^{3(2,3,3)}+G_F^{3(3,3,3)}. \tag{3.36}$$

By careful power counting analysis one can show that only the terms in the first line of (3.36) will contribute pole terms. This simplifies matters considerably, although we note that the third term does include a factor of the t matrix, and thus depends heavily on the space–time under consideration. Let us consider this term first.

From (3.35) and (2.17) we have

$$J^{(1,3,1)}(k, k_0') = \mu^{8-2n} \int \frac{1}{(p^2 - m_-^2)} \frac{1}{[(p+q+k)^2 - m_-^2]} \times \frac{\hat{V}(q_0 - r_0) t_{|\boldsymbol{q}|}(r_0, q_0 + k_0 - k_0')}{(q^2 - m_-^2)(r^2 - m_-^2)[(q+k-k')^2 - m_-^2]} \, \mathrm{d}^n p \, \mathrm{d}^n q \, \mathrm{d}r_0 \tag{3.37}$$

where $r = (r_0, \boldsymbol{q})$, $k' = (k_0', \boldsymbol{k})$. Performing the p integration by introducing a Feynman parameter and using (A.1) of Collins (1974) we find

$$J^{(1,3,1)}(k, k_0') = -\frac{2\mathrm{i}\pi^2}{(n-4)} \int \frac{V(q_0 - r_0) t_{|\boldsymbol{q}|}(r_0, q_0 + k_0 - k_0') \, \mathrm{d}^n q \, \mathrm{d}r_0}{(q^2 - m_-^2)(r^2 - m_-^2)(q+k-k')^2 - m_-^2} + \text{finite}. \tag{3.38}$$

Substituting (3.38) into (3.34) we have, after carrying out the $\mathrm{d}\boldsymbol{k}$ and $\mathrm{d}(k_0 + k_0')$ integrals,

$$\mu^{8-2n} G_{\mathrm{F}}^{3(1,3,1)}(x, x') = \frac{-2\mathrm{i}\Omega^{-2}(\eta)}{16^2 \pi^6 (n-4)} \frac{\delta^n(x - x')}{\sqrt{-g_{x'}}} \times \frac{\int \mathrm{e}^{\mathrm{i}(q_0 - q_0')\eta} \hat{V}(q_0 - r_0) t_{|\boldsymbol{q}|}(r_0, q_0') \, \mathrm{d}^n q \, \mathrm{d}q_0' \, \mathrm{d}r_0}{(q^2 - m_-^2)(r^2 - m_-^2)(q'^2 - m_-^2)} + \text{finite} \tag{3.39}$$

where $q' = (q_0', \boldsymbol{q})$ and we have changed from an integral with respect to $k_0 - k_0'$ to one with respect to $q_0' = q_0 + k_0 - k_0'$. One can already see that this contribution to G_{F}^3 when substituted into (3.36) and then (3.32) gives a pole term which exactly cancels the sum of the two poles involving the t matrix which arise when $\Pi^{(2,1)}$ in (3.31) is added to $\Pi^{(1,2)}$ in (3.29).

Next we consider $J^{(1,1,2)}$, which from (3.34) and (2.17) is

$$J^{(1,1,2)} = -\mu^{8-2n} V(k_0 - k_0') \int \frac{1}{p^2 - m_-^2} \frac{1}{q^2 - m_-^2} \times \frac{1}{(p+q+k)^2 - m_-^2} \frac{1}{(p+q+k')^2 - m_-^2} \, \mathrm{d}^n p \, \mathrm{d}^n q. \tag{3.40}$$

Introducing Feynman parameters and performing the p and q integrals we have

$$\begin{aligned} J^{(1,1,2)} = \mu^{8-2n} \pi^n \Gamma(4-n) \hat{V}(d_0) \iint_0^1 \iint \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma \, \mathrm{d}\rho \, \delta(1 - \alpha - \beta - \gamma - \rho) \\ \times [\alpha\beta + (\alpha+\beta)(\gamma+\rho)]^{4-3n/2} \{m^2[\alpha\beta + (\alpha+\beta)(\gamma+\rho)] - \tfrac{1}{4}s^2 \alpha\beta(\gamma+\rho) \\ - \tfrac{1}{4}d^2[\alpha\beta(\gamma+\rho) + 4(\alpha+\beta)\gamma\rho] + \tfrac{1}{2}ds\alpha\beta(\gamma-\rho)\}^{n-4} \end{aligned} \tag{3.41}$$

where $d \equiv k' - k$, $s = k' + k$. The extraction of the pole terms from the Feynman parameter integral is a little involved and is outlined in the appendix, giving

$$J^{(1,1,2)} = \mu^{8-2n} \pi^n \hat{V}(d_0) \left(\frac{2}{(n-4)^2} + \frac{-1+2\gamma}{n-4} + \frac{2}{n-4} \int_0^1 \ln[m_-^2 - d_0^2 x(1-x)] \, \mathrm{d}x \right) + \text{finite}. \tag{3.42}$$

Substituting (3.42) into (3.34) gives

$$\mu^{8-2n}G_{\mathrm{F}}^{3(1,1,2)}(x,x')$$

$$=\frac{2\Omega^{-2}(\eta)\delta^n(x-x')}{(16\pi^2)^2\sqrt{-g_{x'}}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k_0\eta}\hat{V}(k_0)\left[\frac{1}{(n-4)^2}+\frac{2\gamma-1}{2(n-4)}+\frac{1}{(n-4)}\int_0^1\ln\left(\frac{m_-^2-k_0^2x(1-x)}{4\pi\mu^2\Omega^2(\eta)}\right)\mathrm{d}x\right]\mathrm{d}k_0+\text{finite.} \tag{3.43}$$

We finally obtain the contribution of $J^{(1,1,1)}$, which is given by

$$J^{(1,1,1)}=\mu^{8-2n}\delta(k_0'-k_0)\int\frac{1}{p^2-m_-^2}\,\frac{1}{q^2-m_-^2}\,\frac{1}{(p+q+k)^2-m_-^2}\,\mathrm{d}^np\,\mathrm{d}^nq. \tag{3.44}$$

We can easily obtain the pole term of (3.44) using Collin's (1974) equation (19). Upon substituting this into (3.34) we obtain a contribution

$$\mu^{8-2n}G_{\mathrm{F}}^{3(1,1,1)}(x,x')$$

$$=\frac{1}{2(16\pi^2)^2}\,\frac{1}{(n-4)}[\Omega(\eta)\Omega(\eta')]^{3(2-n)/2}\partial_\mu^x\partial_x^\mu\delta^n(x-x')$$

$$-\frac{6m_-^2\Omega^{-2}(\eta)}{(16\pi^2)^2}\,\frac{\delta^n(x-x')}{\sqrt{-g_{x'}}}$$

$$\times\left[\frac{1}{(n-4)^2}+\frac{\gamma-1}{(n-4)}-\frac{1}{2(n-4)}+\frac{1}{(n-4)}\ln\left(\frac{m_-^2}{4\pi\mu^2\Omega^2(\eta)}\right)\right]+\text{finite.} \tag{3.45}$$

If we note that for an arbitrary scalar test function ψ

$$\left[\Box+\tfrac{1}{4}\left(\frac{n-2}{n-1}\right)R\right]\psi=\Omega^{(-n/2)-1}\partial_\mu\partial^\mu(\Omega^{(n-2)/2}\psi) \tag{3.46}$$

then

$$\int\sqrt{-g_x}\sqrt{-g_{x'}}[\Omega(\eta)\Omega(\eta')]^{3(2-n)/2}[\partial_\mu^x\partial_x^\mu\delta^n(x-x')]\psi(x,x')\,\mathrm{d}^nx\,\mathrm{d}^nx'$$

$$=\int\sqrt{-g_{x'}}\Omega^{(-n/2)-1}(\eta')\delta^n(x-x')\partial_\mu^x\partial_x^\mu(\Omega^{(n-2/2)}(\eta)\psi(x,x'))\,\mathrm{d}^nx\,\mathrm{d}^xx'+\mathrm{O}(n-4)$$

$$=\int\sqrt{-g_x}\left[\Box_x+\tfrac{1}{4}\left(\frac{n-2}{n-1}\right)R(\eta)\right]\psi(x,x)\,\mathrm{d}^nx+\mathrm{O}(n-4) \tag{3.47}$$

which implies

$$[\Omega(\eta)\Omega(\eta')]^{3(2-n)/2}[\partial_\mu^x\partial_x^\mu\delta^n(x-x')]=[\Box_x+\tfrac{1}{6}R(\eta)]\delta^n(x-x')/\sqrt{-g_{x'}}+\mathrm{O}(n-4). \tag{3.47}$$

Using (3.47) in (3.46) and adding the result to (3.43) in the manner dictated by (3.36) we have, after some manipulation using the inverse of (2.15),

$$\mu^{8-2\eta}[G^{3(1,1,1)}(x,x')+3G^{3(1,1,2)}(x,x')$$

$$=\frac{1}{2(16\pi^2)^2}\,\frac{1}{(n-4)}[\Box_x+\tfrac{1}{6}R(\eta)]\frac{\delta^n(x-x')}{\sqrt{-g_{x'}}}$$

$$+\frac{6}{(16\pi^2)^2}\frac{\delta^n(x-x')}{\sqrt{-g_{x'}}}\bigg(-[m^2+(\xi-\tfrac{1}{6})R]\bigg[\frac{1}{(n-4)^2}+\frac{\gamma-1}{n-4}-\frac{1}{2(n-4)}$$
$$+\frac{1}{n-4}\ln\bigg(\frac{m_-^2}{4\pi\mu^2\Omega^2(\eta)}\bigg)\bigg]+\frac{R}{36(n-4)}+\frac{\Omega^{-2}(\eta)}{(n-4)}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k_0\eta}\hat{V}(k_0)$$
$$\times\bigg\{1+\int_0^1\ln\bigg[1-\bigg(\frac{k_0}{m_-}\bigg)^2x(1-x)\bigg]\,\mathrm{d}x\bigg\}\,\mathrm{d}k_0\bigg). \qquad (3.48)$$

To this we add (3.39) multiplied by three and substitute the result in (3.32) to obtain $\Pi^{(2,2)}$. Finally we add $\Pi^{(2,2)}$ to $\Pi^{(2,1)}$, in (3.31), and $\Pi^{(1,2)}$ in (3.29), retaining only the pole terms:

$$\Pi^{\text{pole}}(x,x')=\bigg[\frac{\mathrm{i}\lambda_R^2}{(16\pi^2)^2}\frac{1}{12(n-4)}[\Box_x+\tfrac{1}{6}R(\eta)]-\frac{\mathrm{i}\lambda_R^2}{(16\pi^2)^2}\frac{R}{36(n-4)}$$
$$+\frac{\mathrm{i}\lambda_R^2}{(16\pi^2)^2}[m^2+(\xi-\tfrac{1}{6})R(\eta)]\bigg(\frac{2}{(n-4)^2}+\frac{1}{2(n-4)}\bigg)$$
$$-\frac{\mathrm{i}\lambda_R}{16\pi^2}\frac{1}{(n-4)}[m^2+(\xi-\tfrac{1}{6})R]\bigg]\frac{\delta^n(x-x')}{\sqrt{-g_{x'}}}. \qquad (3.49)$$

We immediately see that all the unwanted non-geometrical infinities have cancelled, leaving only poles that can be absorbed into mass, ξ and 'wavefunction' renormalisations. To see this we write the complete inverse propagator to this order as (see Birrell and Taylor 1978)

$$G'^{-1}(x,x')=\mathrm{i}[\Box_x+\xi_BR+m_B^2]\frac{\delta^n(x-x')}{\sqrt{-g_{x'}}}-\Pi^{\text{pole}}(x,x')-\Pi^{\text{finite}}(x,x')$$
$$=\mathrm{i}\bigg\{\bigg(1-\frac{\lambda_R^2}{(16\pi^2)^2}\frac{1}{12(n-4)}\bigg)\Box_x$$
$$+m^2\bigg[1+\frac{\lambda_R^2}{(n-4)}\bigg(b_{12}-\frac{1}{2(16\pi^2)^2}\bigg)+\frac{\lambda_R^2}{(n-4)^2}\bigg(b_{22}-\frac{2}{(16\pi^2)^2}\bigg)\bigg]$$
$$+R(\eta)\bigg[\xi+\frac{\lambda_R^2}{(n-4)}\bigg(d_{12}-\frac{1}{2(16\pi^2)^2}(\xi-\tfrac{7}{36})\bigg)$$
$$+\frac{\lambda_R^2}{(n-4)^2}\bigg(d_{22}-\frac{2(\xi-\frac{1}{6})}{(16\pi^2)^2}\bigg)\bigg]\bigg\}\frac{\delta^n(x-x')}{\sqrt{-g_{x'}}}+\text{finite}. \qquad (3.50)$$

In obtaining the second equality in (3.50) we have substituted (3.1) to second order for ξ_B and m_B^2. With Z as given in (3.3.), ZG'^{-1} will be analytic at $n=4$ provided we take

$$c_{12}=[12(16\pi^2)^2]^{-1}$$
$$b_{12}=5[12(16\pi^2)^2]^{-1}$$
$$d_{12}=5(\xi-\tfrac{7}{30})[12(16\pi^2)^2]^{-1}$$
$$b_{22}=2(16\pi^2)^{-2}$$
$$d_{22}=2(\xi-\tfrac{1}{6})(16\pi^2)^{-2}. \qquad (3.51)$$

Once again the results obtained for b_{12}, b_{22} and c_{12} are the same as obtained by Collins (1974) in flat space–time. It is also worth noting that even in the 'conformally invariant' case in which $m = 0$, $\xi = \frac{1}{6}$, it is necessary to perform a renormalisation of ξ. That is, $\xi = \frac{1}{6}$ and $m = 0$ does not imply $\xi_B = \frac{1}{6}$, even though it does imply $m_B = 0$. Thus if the bare Lagrangian is conformally invariant the renormalised one will not be. The reason for this is simple; the extension to n dimensions breaks the conformal invariance, since a Lagrangian with $\xi = \frac{1}{6}$ and $m = 0$ is only conformally invariant for $n = 4$. If we had extended in such a way as to maintain conformal invariance, then both the bare and renormalised Lagrangians would be conformally invariant. An example of a particular calculation using an extension scheme which maintains conformal invariance in n dimensions is found in the paper of Drummond (1975) who considers massless $\lambda\phi^4$ to third order in spherical space–time. Drummond notes that the curvature does not induce any divergent mass terms (R = constant) in the Lagrangian. Had he used a scheme which did not extend conformally then such terms would have arisen as we have found above. The difference between the two results is no more than the usual renormalisation ambiguity and does not affect physical results.

4. Conclusion

It has been shown that the momentum space technique for curved space–time quantum field theory calculations (Birrell 1979a) is well suited to calculations involving interacting fields. In particular, it allows the use of dimensional regularisation and Feynman parameter techniques for the calculation of the infinite parts of S-matrix elements exactly as in flat space–time. In addition, once these infinite parts have been renormalised away it offers powerful tools for the calculation of finite parts of physically interesting amplitudes. Admittedly the calculation of all the finite parts to second order in ϕ^4 theory would be an enormous undertaking for most non-trivial space–times. However it is perfectly feasible to consider a calculation to first order, and from preliminary analysis (Birrell and Ford 1978) we expect important contributions to cosmological particle production (for example) from contributions at this order. More detailed studies of first-order finite parts are underway (Birrell *et al* 1980).

The calculation in the previous section is of importance in its own right, as it gives independent confirmation to the statement of Bunch *et al* (1979) that all the state-dependent infinities cancel to second order. It is not clear that this cancellation will continue to higher orders (see Birrell and Taylor (1978) for a detailed discussion).

The subject of interacting quantum field theory in curved space–time thus has many topics of interest to be considered, both of a fundamental nature and in its application to cosmology and astrophysics. In both areas the momentum space method should provide a useful means for further development.

Acknowledgments

The author wishes to thank Drs P C W Davies and L H Ford and Professor J G Taylor for numerous interesting discussions on the subject of interacting quantum fields in curved backgrounds.

This work was supported by a Flinders University of South Australia Overseas Scholarship.

Appendix

We consider the extraction of the pole term from the integral (3.41). The method used is similar to that used by Collins (1974, appendix B) for the extraction of the pole in the integral occurring in (3.43).

The integral in (3.41) can possibly give rise to poles from the regions of the range of integration where $\alpha\beta+(\alpha+\beta)(\gamma+\rho)=0$. These regions are given by $\alpha=0$, $\beta=0$; $\alpha=1$, $\beta=0$; $\alpha=0$, $\beta=1$; in each case γ, ρ are constrained by the limits of integration and the delta function.

We first obtain the pole from $\alpha=0$, $\beta=0$ by making the change of variables

$$\alpha=xy \qquad \beta=x(1-y) \qquad \gamma=z(1-x) \qquad \rho=(1-x)(1-z) \tag{A1}$$

and we consider the ρ integral to have been removed using the delta function. Now the singularity at $\alpha+\beta=0$ lies only in the x integration, as $x\to0$, for then $\alpha\beta+(\alpha+\beta)(\gamma+\rho)\sim x$ and

$$J^{(1,1,2)}\sim\pi^n\Gamma(4-n)\hat{V}(d_0)\iint_0^1\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,x x^{4-3n/2}[m^2x-d^2xz(1-z)]^{n-4} \tag{A2}$$

where we have also used the fact that the Jacobian of the transformation is $x(1-x)\sim x$ as $x\to0$. The x and y integrations can be performed and the pole extracted, giving

$$J^{(1,1,2)}\sim\pi^n\hat{V}(d_0)\left(\frac{1}{n-4}+\gamma\right)\frac{2}{n-4}\left[1+(n-4)\int_0^1\ln[m^2-d^2z(1-z)]\,\mathrm{d}z\right]. \tag{A3}$$

Next consider the pole at $\alpha=0$, $\beta=1$ which, by symmetry is equal to the pole at $\beta=0$, $\alpha=1$. Making a change of variables

$$\alpha=x(1-xz) \qquad \beta=(1-x)(1-xz) \qquad \gamma=xyz \qquad \rho=zx(1-y) \tag{A4}$$

which isolates the possible singularity to the x integral, since then

$$\alpha\beta+(\alpha+\beta)(\gamma+\rho)\sim x(1+z) \qquad \text{as } x\to0$$

we have

$$J^{(1,1,2)}\sim\pi^n\Gamma(4-n)\hat{V}(d_0)\int_0^1\mathrm{d}z\int_0^1\mathrm{d}x\,zx^{2-n/2}(1+z)^{-n/2}(m^2)^{n-4}. \tag{A5}$$

The integral is (A5) gives no pole.

There is still, of course, a pole arising from the pole in the gamma function multiplying the finite part of the integral in $J^{(1,1,2)}$. To calculate the finite part of the integral in (3.41) we subtract from it the integral in (A2) and let $n\to4$. This gives

$$\begin{aligned}\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\int_0^1\mathrm{d}z\,x\left(\frac{1-x}{x^2\{x[y(1-y)-1]+1\}^2}-\frac{1}{x^2}\right)\\ =-\int_0^1\mathrm{d}y[1+\ln(y(1-y))]=1.\end{aligned} \tag{A6}$$

Thus to (A3) is added the pole part of $\pi^n\Gamma(4-n)\hat{V}(d_0)\times1$ which is $-\pi^n\hat{V}(d_0)(n-4)^{-1}$. Doing this, noting that $d^2=d_0^2$, and changing integration variable z to x gives (3.42).

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